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On the Property of Monotonic Convergence for Beta Operators*

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We show that beta operators satisfy the property of monotonic convergence under convexity. This gives a positive answer to a question recently posed by M. K. Khan. Some additional properties, consequences and applications are also discussed. Throughout this paper, probabilistic methods play a fundamental role.

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1. Introduction

Let B_t be the beta operator defined by

$$B_t(f, x) := \int_0^1 f(\theta) \frac{\theta^{tx-1}(1-\theta)^{t(1-x)-1}}{B(tx, t(1-x))} d\theta, \qquad t > 0, \qquad x \in (0, 1),$$

where $B(\cdot, \cdot)$ is the beta function and f is any real measurable function defined on $(0, 1)$ such that $B_t(|f|, x) < \infty$. If f is defined on $[0, 1]$, we set

$$B_t(f, i) := f(i), \qquad i = 0, 1. \tag{1}$$

This operator has been considered by several authors (see, for instance, [1, 8, 12, 15]). A slight modification of B_t is the operator B_t^* given by

$$B_t^*(f, x) := \int_0^1 f(\theta) \frac{\theta^{tx}(1-\theta)^{t(1-x)}}{B(tx+1, t(1-x)+1)} d\theta, \qquad t \geqslant 0, \qquad x \in [0, 1],$$

which, for natural values of the parameter t, has been introduced by Lupaş in [14]. A significant difference between B_t and B_t^* is that B_t reproduces linear functions, whereas B_t^* does not. The operator B_t is also quite different from the (double indexed) beta operator studied by Upreti [18].

It is well known that

$$B_t(f, x) \to f(x), \qquad x \in (0, 1),$$

as $t \to \infty$, whenever f is a real continuous bounded function defined on $(0, 1)$ (cf. [8, 12, 15]). In this paper, attention is focussed on some properties of the "approximation path", i.e., on some properties of the function

$$p(t) = p_{x,f}(t) := B_t(f, x), \qquad t > 0,$$

where f and x are fixed.

Firstly, we show that if f is convex then $p(\cdot)$ is nonincreasing. More precisely, we show the following theorem which gives a positive answer to a question posed by M. K. Khan in [12].

Theorem 1. *Let $x \in (0, 1)$ and $0 < r < t$. If f is a convex function defined on $(0, 1)$ such that $B_s(|f|, x) < \infty$, for $s = r, t$, then*

$$B_r(f, x) \geqslant B_t(f, x).$$

As far as positive linear operators are concerned, there are two standard ways of showing the property of monotonic convergence under convexity: the purely analytical approach based on computations which depend heavily on the particular form of the operator considered, and the probabilistic approach introduced by R. A. Khan [13] based on the conditional version of Jensen's inequality (see also [2, 4]). In the case at hand, neither of these two standard methods seem to provide a simple proof of Theorem 1 (see section 3 for a more detailed discussion).

In sections 2 and 4, we give two different proofs of Theorem 1. Both proofs use probabilistic methods as fundamental tools. However, each of them emphasizes a particular feature of B_t which, on the other hand, may be of interest in order to understand the structure and the behaviour of this operator with respect to other properties.

Secondly, we consider absolutely and completely monotone functions defined on $[0, 1]$. For these special cases of convex functions we obtain the following result.

THEOREM 2. *If $f \in C[0, 1]$ is an absolutely or completely monotone function, then, for any $x \in (0, 1)$, the approximation path*

$$p(t) := B_t(f, x), \qquad t \geqslant 0,$$

is a completely monotone function on $[0, \infty)$.

The proof given in section 5 is based upon the representation of an absolutely monotone function defined on $[0, 1]$ as the probability generating function of some nonnegative, integer-valued random variable.

Finally, Theorems 1 and 2 have interesting consequences, some of them concerning other well-known Bernstein-type operators. They are discussed in section 6.

2. First Proof of Theorem 1

To begin with, we give a suitable probabilistic representation for B_t in terms of gamma processes. We recall that a gamma process is a stochastic process $\{U_t : t \geqslant 0\}$ starting at the origin, having stationary independent increments and such that, for each $t > 0$, U_t has the gamma distribution with density

$$d_t(\theta) := \frac{\theta^{t-1} e^{-\theta}}{\Gamma(t)}, \qquad \theta > 0.$$

Let $\{U_t : t \geqslant 0\}$ and $\{V_t : t \geqslant 0\}$ be two independent gamma processes defined on the same probability space. Then, for every $t > 0$ and $x \in (0, 1)$, the random variable

$$Z_t^x := \frac{U_{tx}}{U_{tx} + V_{t(1-x)}}, \tag{2}$$

has the beta distribution with parameters tx, $t(1-x)$ and, therefore, we can write

$$B_t(f, x) = Ef(Z_t^x), \tag{3}$$

where E denotes mathematical expectation. Observe that (3) is consistent with (1).

Under the assumptions in Theorem 1, we claim that

$$E((U_{rx} + V_{r(1-x)})\, f(Z_r^x) \mid \mathscr{H}_t) \geqslant \frac{r}{t}\, (U_{tx} + V_{t(1-x)})\, f(Z_t^x) \qquad \text{a.s.}, \tag{4}$$

where $E(\cdot \mid \cdot)$ denotes conditional expectation and $\mathscr{H}_t$ is the σ-algebra generated by U_{tx} and $V_{t(1-x)}$. In order to see this, note that $U_{rx} + V_{r(1-x)}$ and $f(Z_r^x)$ are independent (cf. [10]) and therefore

$$E\, |(U_{rx} + V_{r(1-x)})\, f(Z_r^x)| = E(U_{rx} + V_{r(1-x)})\, E\, |f(Z_r^x)| < \infty,$$

so that the left-hand-side in (4) is well defined. Since f is convex, we can find sequences of real numbers $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ such that

$$f(z) = \sup_{n \geqslant 1}\, [a_n z + b_n], \qquad z \in (0, 1), \tag{5}$$

(cf. [5, Th. 7.3.4]). Then

$$\begin{aligned} E((U_{rx} + V_{r(1-x)})\, f(Z_r^x) \mid \mathscr{H}_t) \geqslant \sup_{n \geqslant 1}\, [&a_n E(U_{rx} \mid \mathscr{H}_t) \\ &+ b_n E(U_{rx} + V_{r(1-x)} \mid \mathscr{H}_t)] \qquad \text{a.s.} \end{aligned} \tag{6}$$

Since $\{U_t : t \geqslant 0\}$ and $\{V_t : t \geqslant 0\}$ are mutually independent, and the random variables U_{rx}/U_{tx} and $U_{tx} = U_{rx} + (U_{tx} - U_{rx})$ are also independent (cf. [10]), we have

$$\begin{aligned} E(U_{rx} \mid \mathscr{H}_t) = E(U_{rx} \mid U_{tx}) &= U_{tx}\, E\left(\frac{U_{rx}}{U_{tx}} \,\middle|\, U_{tx}\right) \\ &= U_{tx}\, E\left(\frac{U_{rx}}{U_{tx}}\right) = \frac{r}{t}\, U_{tx} \qquad \text{a.s.} \end{aligned} \tag{7}$$

Similarly,

$$E(V_{r(1-x)} \mid \mathscr{H}_t) = \frac{r}{t} V_{t(1-x)} \qquad \text{a.s.} \tag{8}$$

and the claim follows from (5)–(8). To complete the proof of Theorem 1, it suffices to take mathematical expectations in (4), recalling that, for $s = r, t$, the random variables $U_{sx} + V_{s(1-x)}$ and $f(Z_s^x)$ are independent.

3. Discussion

The standard probabilistic technique mentioned in the introduction can be described as follows:

Assume that the family of operators $(L_t)_{t>0}$ allows for a representation having the form

$$L_t(f, x) = Ef(Z_t^x), \qquad x \in I, \qquad t > 0, \tag{9}$$

where I is an interval of the real line, $\{Z_t^x : t > 0\}$ is a stochastic process taking values in I, and f is any real function defined on I and satisfying appropriate integrability conditions. Assume, further, that the stochastic process satisfies the martingale-type condition

$$E(Z_r^x \mid Z_t^x) = Z_t^x \qquad \text{a.s.,} \qquad 0 < r < t. \tag{10}$$

Then, if f is a convex function, we have by (10) and the conditional version of Jensen's inequality (cf. [6])

$$Ef(Z_r^x) = E(E(f(Z_r^x) \mid Z_t^x)) \geqslant E(f(E(Z_r^x \mid Z_t^x))) = Ef(Z_t^x),$$

that is, by (9),

$$L_r(f, x) \geqslant L_t(f, x). \tag{11}$$

In other words, to show (11) it suffices to find a stochastic process such that both (9) and (10) hold true.

However, the preceding argument does not apply to the process $\{Z_t^x : t > 0\}$ defined in (2), because (10) is not satisfied. Actually, it is not hard to see that, for $0 < r < t$,

$$E(Z_t^x \mid Z_r^x) = \frac{r}{t} Z_r^x + \frac{t-r}{t} x \qquad \text{a.s.,}$$

which implies

$$E(Z_r^x Z_t^x) = E(Z_r^x E(Z_t^x \mid Z_r^x)) = \frac{r(rx+1)\,x}{t(r+1)} + \frac{(t-r)\,x^2}{t}. \tag{12}$$

If (10) were true, we would obtain

$$E(Z_r^x Z_t^x) = E(Z_t^x E(Z_r^x \mid Z_t^x)) = E(Z_t^x)^2 = \frac{tx+1}{t+1}\,x,$$

which contradicts (12).

On the other hand, we can also write

$$B_t(f, x) = Ef\left(\frac{U_{tx}}{U_t}\right), \tag{13}$$

where $\{U_t : t \geqslant 0\}$ is a gamma process. This probabilistic representation is quite different from (3) and was introduced in [1] to study preservation of monotonicity, convexity and Lipschitz constants. Since, for $0 < x < 1$ and $0 < r \leqslant tx$, the random variables U_{rx}/U_r and U_{tx}/U_t are independent (cf. [10]), we have

$$E\left(\frac{U_{rx}}{U_r}\middle|\frac{U_{tx}}{U_t}\right) = x \qquad \text{a.s.} \qquad 0 < x < 1, \qquad 0 < r \leqslant tx.$$

Thus, neither does this representation satisfy (10).

We therefore conclude that the crucial point in the first proof of Theorem 1 is the submartingale-type property (4). With regard to B_t, this property plays the same role as property (10) does with regard to other usual approximation operators.

4. Second proof of Theorem 1

The main point in this proof consists in relating B_t with the operator H_t defined below, which allows for a probabilistic representation fulfilling condition (10).

Consider the beta-type operator H_t given by

$$H_t(g, x) := \frac{1}{B(tx/1+x, (t/1+x)+1)} \times \int_0^\infty g(u)\, \frac{u^{(tx/1+x)-1}}{(1+u)^{t+1}}\, du, \qquad t, x > 0,$$

where g is any real measurable function on $(0, \infty)$ such that $H_t(|g|, x) < \infty$. We also define the following positive linear operators

$$S(g, x) := (1-x)\, g(x/1-x), \qquad x \in (0, 1),$$

$$T(f, x) := (1+x)\, f(x/1+x), \qquad x \in (0, \infty),$$

where g (resp. f) is any real function defined on $(0, \infty)$ (resp. $(0, 1)$). Elementary calculations show that

$$B_t(f, x) = S \circ H_t \circ T(f, x), \tag{14}$$

where '$\circ$' denotes composition, whenever one of the two sides is well defined. It is easy to check that the operator T preserves convexity. Therefore, Theorem 1 is a consequence of the following Lemma.

LEMMA 1. *Let $x > 0$ and $0 < r < t$. If g is a convex function defined on $(0, \infty)$ such that $H_s(|g|, x) < \infty$, for $s = r, t$, then*

$$H_r(g, x) \geqslant H_t(g, x).$$

Proof of Lemma 1. We have the following probabilistic representation for H_t:

$$H_t(g, x) = Eg\left(\frac{U_{tx/1+x}}{V_{(t/1+x)+1}}\right),$$

where $\{U_s : s \geqslant 0\}$ and $\{V_s : s \geqslant 0\}$ are two independent gamma processes defined on the same probability space. Using the same procedure as in the proof of Lemma 2 (a) in [3], we obtain for $t > r > 0$

$$E\left(\frac{U_{rx/1+x}}{V_{(r/1+x)+1}} \,\middle|\, U_{tx/1+x}, V_{(t/1+x)+1}\right) = \frac{U_{tx/1+x}}{V_{(t/1+x)+1}} \qquad \text{a.s.},$$

and the conclusion follows by the standard argument mentioned at the beginning of the preceding section.

Remark 1. Observe that

$$T \circ S = I_*, \qquad S \circ T = I^*,$$

where I_* and I^* are the identity operators in the corresponding function spaces. We also have

$$H_t(g, x) = T \circ B_t \circ S(g, x),$$

which is the converse relation of (14). Since the operator S preserves convexity, the statements in Theorem 1 and Lemma 1 are, in fact, equivalent.

5. Proof of Theorem 2

We shall start by extending in a natural way the definition of $B_t(f, x)$ to the whole strip $\{(x, t): 0 \leqslant x \leqslant 1,\ t \geqslant 0\}$, whenever $f \in C[0, 1]$. Let $x \in [0, 1]$, $k = 1, 2, ...,$ and let Z_t^x be the random variable defined in (2). Since

$$E(Z_t^x)^k = \prod_{j=0}^{k-1} \frac{tx + j}{t + j} \to x, \qquad (t \to 0),$$

the method of moments guarantees that Z_t^x converges weakly, as $t \to 0$, to the Bernoulli distribution with parameter x (cf. [6]). Therefore, if $f \in C[0, 1]$, we have by (3) and the Helly-Bray theorem

$$B_t(f, x) \to (1 - x) f(0) + xf(1), \qquad (t \to 0).$$

In other words, we may define by continuity

$$B_0(f, x) := (1 - x) f(0) + xf(1), \qquad x \in [0, 1]. \tag{15}$$

This was implicitely assumed in the statement of Theorem 2.

Suppose, firstly, that $f \in C[0, 1]$ is absolutely monotone. Without loss of generality, we can assume that $f(1) = 1$. Consequently (cf. [9]), f is the probability generating function of a nonnegative, integer-valued random variable N, i.e.,

$$f(\theta) = E\theta^N, \qquad 0 \leqslant \theta \leqslant 1.$$

If $P(N > 1) = 0$, the conclusion is trivial. Assume, then, that $P(N > 1) > 0$. We have, for $x \in (0, 1)$,

$$p(t) = \sum_{n=0}^{\infty} P(N = n)\, E(Z_t^x)^n = P(N = 0) + xP(N = 1) + xP(N > 1)\, \phi(t),$$

where

$$\phi(t) := \sum_{n=1}^{\infty} \frac{P(N = n + 1)}{P(N > 1)} \prod_{k=1}^{n} \phi_k(t), \qquad t \geqslant 0,$$
$$\phi_k(t) := \frac{tx + k}{t + k} = x + (1 - x) \frac{k}{t + k}, \qquad k = 1, 2, ... \tag{16}$$

We claim that ϕ is the Laplace transform of a nonnegative random variable Y, i.e.,

$$\phi(t) = Ee^{-tY}, \qquad t \geqslant 0.$$

As a consequence (cf. [9]), ϕ, and therefore p, are both completely monotone functions. To show the claim, let $M, X_1, X_2, \ldots$ be mutually independent random variables such that M has the distribution given by

$$P(M = n) := \frac{P(N = n+1)}{P(N > 1)}, \qquad n = 1, 2, \ldots,$$

and each X_k has the distribution

$$x\nu_0 + (1-x)\,\nu_k,$$

where ν_0 is the unit mass at 0 and ν_k is the exponential distribution with parameter k. Define

$$Y := \sum_{n=1}^{M} X_n.$$

In view of (16), we have for $t \geqslant 0$,

$$Ee^{-tY} = \sum_{n=1}^{\infty} P(M = n) \prod_{k=1}^{n} Ee^{-tX_k} = \sum_{n=1}^{\infty} \frac{P(N = n+1)}{P(N > 1)} \prod_{k=1}^{n} \phi_k(t) = \phi(t),$$

and the claim is shown.

Finally, if $f \in C[0, 1]$ is completely monotone, the conclusion follows from the equality

$$B_t(f, x) = B_t(f(1-\theta), 1-x), \qquad t \geqslant 0, \qquad x \in [0, 1]$$

and the fact that $f(1-\theta)$ is absolutely monotone on $[0, 1]$.

6. Consequences and Applications

(A) *Lupaş Beta Operators*

If $x \in [0, 1]$ and f is a real measurable function defined on $(0, 1)$, we can write

$$B_t^*(f, x) = B_{t+2}\left(f, \frac{tx+1}{t+2}\right), \qquad t \geqslant 0, \tag{17}$$

whenever one of the two sides in (17) is well defined. In particular, for $x=\frac{1}{2}$, we have

$$B_t^*(f, \tfrac{1}{2}) = B_{t+2}(f, \tfrac{1}{2}), \qquad t \geqslant 0$$

and, therefore, under obvious assumptions, the property of monotonic convergence holds in this case. For $x \neq \frac{1}{2}$, the property may fail (take, for instance, $f(\theta)=\theta$ or $f(\theta)=1-\theta$). Notwithstanding, in view of (17) and taking into account that B_t preserves monotonicity (cf. [1]), we deduce from Theorem 1:

Corollary 1. *Let f be a convex function defined on $(0, 1)$ such that $B_r^*(|f|, x) < \infty$ and $B_t^*(|f|, x) < \infty$, for $t > r > 0$. Then*

$$B_r^*(f, x) \geqslant B_t^*(f, x),$$

if one of the two following conditions is fulfilled:

(a) *f is nondecreasing and $x \in [0, \frac{1}{2}]$.*

(b) *f is nonincreasing and $x \in [\frac{1}{2}, 1]$.*

(B) *Inverse Beta Operators*

Let us define

$$T_t(f, x) := \int_0^\infty f(\theta) \frac{1}{B(tx, t)} \frac{\theta^{tx-1}}{(1+\theta)^{tx+t}}\, d\theta, \qquad t>0, \qquad x>0,$$

where f is any real measurable function defined on $(0, \infty)$ such that $T_t(|f|, x) < \infty$. If f is defined on $[0, \infty)$, we set $T_t(f, 0) := f(0)$. This operator has been introduced and studied in [3] (see also [1]). It is clear that

$$T_t(f, x) = B_{t(x+1)}\left(f\left(\frac{\theta}{1-\theta}\right), \frac{x}{1+x}\right), \qquad t>0, \qquad x>0, \qquad (18)$$

provided that one of the two sides in (18) is well defined, and, therefore, we have from Theorem 1:

Corollary 2. *Let $x>0$, $t>r>0$ and let f be a real measurable function defined on $(0, \infty)$ such that $T_s(|f|, x) < \infty$, for $s = r, t$. If $f(\theta/(1-\theta))$ is convex on $(0, 1)$, then*

$$T_r(f, x) \geqslant T_t(f, x).$$

Remark 2. Observe that $f(\theta/(1-\theta))$ is convex on $(0, 1)$ if either $f(\theta)$ or $f(1/\theta)$ is convex and nondecreasing on $(0, \infty)$. Thus, Corollary 2 extends the results given in [3, Sect. 3]. On the other hand, let $f(\theta)$ be a real continuous function defined on $[0, \infty)$, having a finite limit as $\theta \to \infty$. From (18) and Theorem 2, we deduce the following: If $f(\theta/(1-\theta))$ is absolutely or completely monotone on $[0, 1]$, then, for any $x > 0$, the approximation path

$$p(t) := T_t(f, x), \qquad t \geqslant 0,$$

is a completely monotone function on $[0, \infty)$. In general, the complete monotonicity of f does not imply the complete monotonicity of p. A counterexample can be found in [3, Remark 1].

(C) *Stancu Operators and Generalized Bleimann–Butzer–Hahn Operators*

Stancu operators P_n^α are defined (cf. [8, 15–17]) by

$$P_n^\alpha(f, x) := B_{\alpha^{-1}}(P_n f, x), \qquad x \in [0, 1], \qquad \alpha > 0, \qquad n = 1, 2, ..., \tag{19}$$

where f is any real function defined on $[0, 1]$ and P_n is the Bernstein operator, i.e.,

$$P_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} \theta^k (1-\theta)^{n-k}, \qquad \theta \in [0, 1], \qquad n = 1, 2, ...$$

Since P_n preserves convexity, it follows from (19) and Theorem 1 that $P_n^\alpha f$ decreases to $P_n f$, as α decreases to 0, whenever f is a convex function on $[0, 1]$.

Similarly, generalized Bleimann–Butzer–Hahn operators L_n^α (cf. [1, 3]) are defined by

$$L_n^\alpha(f, x) := T_{\alpha^{-1}}(L_n f, x), \qquad x \geqslant 0, \qquad \alpha > 0, \qquad n = 1, 2, ..., \tag{20}$$

where f is any real function defined on $[0, \infty)$ and L_n is the operator introduced by Bleimann, Butzer and Hahn in [7], i.e.,

$$L_n(f, \theta) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k} \frac{\theta^k}{(1+\theta)^n}, \qquad \theta \geqslant 0, \qquad n = 1, 2, ...,$$

For $x > 0$ and $n = 1, 2, ...$, we obtain from (20) and Corollary 2

$$L_n^{\alpha_1}(f, x) \geqslant L_n^{\alpha_2}(f, x), \qquad \alpha_1 > \alpha_2 > 0, \tag{21}$$

whenever $L_n(f, \theta/(1-\theta))$ is a convex function on $(0, 1)$. Since

$$L_n\left(f, \frac{\theta}{1-\theta}\right) = P_n\left(f\left(\frac{nz}{n+1-nz}\right), \theta\right), \qquad \theta \in (0, 1),$$

the inequality in (21) holds if either $f(\theta)$ or $f(1/\theta)$ is convex and non-decreasing on $(0, \infty)$. This extends some results concerning L_n^α contained in [3, Sect. 3].

(D) *Other Applications*

Some interesting applications may be obtained by applying Theorem 1 to particular functions. Take, for instance, $f(x) = x^p$, $(p > 0)$. Then, for $p \geqslant 1$ (resp. $0 < p < 1$) and $x \in (0, 1)$, we can assert that the quantity

$$\frac{\Gamma(tx+p)\,\Gamma(t)}{\Gamma(tx)\,\Gamma(t+p)}$$

decreases (resp. increases) to x^p, as t increases to ∞, and, in view of (15), it increases (resp. decreases) to x, as t decreases to 0.

Finally, some inequalities concerning convex polynomials and binomial coefficients are given in the following

PROPOSITION 1. *Let* $Q(x) := \sum_{k=2}^m a_k x^k$, $(m \geqslant 2)$, *be a convex polynomial on* $[0, 1]$. *Then*

$$\text{(a)} \qquad \sum_{k=2}^m a_k \binom{t+k-1}{k-1}^{-1} \sum_{j=1}^{k-1} \frac{1}{t+j} \geqslant 0, \qquad \textit{for all} \quad t > 0.$$

$$\text{(b)} \qquad \sum_{k=2}^m a_k \binom{n+k}{k}^{-1} \frac{k-1}{k} \geqslant 0, \qquad n = 1, 2, \ldots$$

Proof. (a) We have

$$B_t(Q, x) = x g(t, x), \qquad t > 0, \qquad x \in [0, 1],$$

where

$$g(t, x) := \sum_{k=2}^m a_k \prod_{j=1}^{k-1} \frac{tx+j}{t+j}.$$

Theorem 1, together with an elementary continuity argument, implies that the function

$$g_0(t) := g(t, 0), \qquad t > 0,$$

is nonincreasing. The inequality in (a) merely says that

$$g_0'(t) \leqslant 0, \qquad t > 0.$$

Similarly, (b) follows from the inequality

$$B_n(Q, x) - B_{n+1}(Q, x) \geqslant 0, \qquad x \in [0, 1], \qquad n = 1, 2, \ldots$$

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